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# POTENTIAL VORTICITY AND THE ELECTROSTATICS ANALOGY: ERTEL-ROSSBY FORMULATION

by

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Abstract

The extension, to the Ertel-Rossby PV, of earlier ideas concerning the isomorphism between the theory of electrostatics and the quasi-geostrophic potential vorticity (PV) is given. Anomalies of PV substance, defined relative to an arbitrary zonal-mean or horizontal-average flow, are described in terms of the divergence of a vector field. This vector is the sum of a linear and a non-linear contribution and can be written as a dielectric tensor acting on the geopotential gradient. It is found that the linear components of the tensor differ from those for the quasi-geostrophic PV only if there exists a vertical variation of background PV, such as occurs at the tropopause, or if there is shear of the assumed background flow. The non-linear components are zero for the quasi-geostrophic case.

The forms of free, bound and total charge are defined for highly accurate non-linear forms of the PV. The free space Green's function for the operator defining the total charge is identical to that for quasi-geostrophic theory and provides a scheme whereby the field attributed to each PV element is an invariant quantity. One of the most important results arising from this formulation is that the non-linearities in the definition of PV can be neglected when considering the far-field effect of PV anomalies. A simple example of these ideas is given analytically for a uniform anomaly of semi-geostrophic PV embedded in an otherwise uniform background PV.

The dielectric constant and bound charge are calculated and give a clear insight into the differences between this and the quasi-geostrophic solution. There is a parallel between this local modification of the far-field solution and that which occurs in quantum field theory.

## 1. Introduction

In Bishop and Thorpe 1994 (hereafter BT) the isomorphism between a quasi-geostrophic potential vorticity anomaly and a static electrical charge was described. This gave a convenient picture of a PV anomaly being associated with a field distributed through space; the anomaly being observable as the divergence of the field. Outside the anomaly the field is non-divergent. In this paper we consider the extension of these ideas to the Ertel-Rossby form of the potential vorticity (PV). The PV can of course be written in a form proportional to the divergence of a vector field: $PV = \frac{1}{\rho} \nabla.( \zeta \theta )$. The vector field in question is the product of the three-dimensional vorticity vector and the potential temperature. The divergence form of the Ertel-Rossby PV is fundamental in that the mass-weighted volume integral of PV only depends on the normal component of the vector field at the boundaries. This property is a familiar one and is, of course, common with the quasi-geostrophic form and electrical charges and their fields.

There is, in principle, a substantial difference between the quasi-geostrophic (q) and the full form for the potential vorticity (PV). The former is a linear quantity which can be interpreted as a modification of the vertical component of absolute vorticity. On the other hand the PV is a quadratic quantity with contributions from the product of vertical component of vorticity and static stability. One might conclude therefore that there was no electrostatics analogy for PV but this proves not to be the case.

Here it will be shown that the field close to a PV anomaly is composed of a linear part plus a non-linear component due to the existence of additional PV charge interpretable in terms of bound charge as introduced by BT. The electrostatics analogy helps to bridge the conceptual gap that exists in interpreting effects due to

anomalies of quasi-geostrophic and Ertel-Rossby potential vorticity. It is known in quantum mechanics, Bjorken and Grell 1964, that very close to an electric charge Coulomb's law is modified to include the effect of a so-called vacuum polarization. This latter is a quantum effect best visualised in terms of this apparently contradictory notion of a polarizability of the vacuum.

In this paper, we show that the electrostatics analogy applies to highly accurate estimates of the PV such as those based on non-linear balance. In section 2 the general theoretical framework is established. Results are then summarized in section 3 in the simple case of geostrophic flow on an f-plane (section 2 can be omitted by readers content with the summary of section 3). In section 4 a comparison is made with the quasi-geostrophic formulation given in BT. The symmetries of the semi-geostrophic system permit elegant analytic solutions, presented in section 5, from which precise descriptions of non-linear bound charge can be made.

The motivation for this research is to cast light on the problem of attribution whereby the flow and temperature associated with particular PV anomalies is sought. This is required within the framework of piecewise PV inversion as discussed by Davis and Emanuel 1991. Further discussion on the notion of attribution can be found in BT and in Bishop 1994.

## 2. PV and electrostatics - beyond quasi-geostrophy

The purpose of this section is to demonstrate the analogy between electrostatics and PV for forms of the PV which are considerably more accurate than the quasi-geostrophic form of the PV. As we shall show, an analogy to electrostatics can be made whenever the Ertel-Rossby PV can be accurately appoximated with the form,

$$\rho\, PV \;=\; \frac{\theta_0}{g}\;\nabla.\left[-\frac{\partial^2\psi}{\partial x\partial z}\,\frac{\partial\phi}{\partial z},\; -\frac{\partial^2\psi}{\partial y\partial z}\,\frac{\partial\phi}{\partial z},\; \left(f+\nabla_h^2\psi\right)\frac{\partial\phi}{\partial z}\right] \qquad (1)$$

where $\phi$ is the geopotential and $\psi$ is the streamfunction for a non-divergent part of the flow, $\rho$ is density and $\theta_0$ is a constant potential temperature. As discussed by Xu (1994), this type of approximation is accurate whenever the rate of change of the

vorticity associated with the divergent part of the flow together with its advection, tilting and stretching, can be neglected. This is a good approximation for synoptic scale systems. Xu argues that smaller scale meteorological phenomena such as fronts, curved fronts and some meso-scale vortices should also be well represented by this type of approximation. Raymond and Jiang (1990) and Raymond (1992) have shown, by direct simulation, that models which use PV in the form given by (1) can produce realistic simulations of some types of meso-scale convective systems.

To describe anomalies of PV it is necessary to subtract from the local PV an appropriate background distribution - the whole notion of an anomaly depends on this. Here we define a geostrophically balanced basic state flow dependent on latitude, y, and height, z, in an arbitrary way: $\phi = \bar{\phi}(y,z) + \phi'(x,y,z,t)$ and $\psi = \bar{\psi}(y,z) + \psi'(x,y,z,t)$. The basic state mass weighted PV is then described by

$$\rho\ \overline{PV} = \frac{\theta_0}{g}\ \nabla.\left[\ 0,\ -\frac{\partial^2\bar{\psi}}{\partial y \partial z}\ \frac{\partial\bar{\phi}}{\partial z},\ \bar{\zeta}\ \frac{\partial\bar{\phi}}{\partial z}\ \right] \qquad (2)$$

where $\bar{\zeta} = f + \dfrac{\partial^2\bar{\psi}}{\partial y^2}$.

To proceed it is convenient to define a reference mass-weighted potential vorticity, $\rho_0\ PV_0 = f_0\ N_0^2\ \theta_0/g$, where $N_0^2$ and $\rho_0$ are a constant reference static stability and density respectively. Equation (1) may then be rewritten in the form:

$$e_f' = f_0\ \rho\ (PV - \overline{PV})/\ (\rho_0\ PV_0) \qquad (3)$$

Note that it is the mass-weighted potential vorticity, $\rho$ PV, that appears in this definition. This is referred to as the amount of PV substance per unit volume by Haynes and McIntyre (1991) and hence satisfies the conservation properties mentioned there. It is entirely consistent with Haynes and McIntyre's definition of PV substance to refer to $e_f'$ as the free PV charge density. The subscript f indicates that this part of the PV field is analogous to the free charge in electrostatics; i.e. it is this part of the mass-weighted PV field that can be moved by advective and other fluxes along isentropes; cf Haynes and McIntyre (1991). The definition for the potential

vorticity given by equation (3) is in accord with that of Rossby (1940) who referred the PV to a standard latitude and static stability. It has the aesthetic advantage of having the dimensions of vorticity and being proportional to the perturbation PV substance. To calculate this form of potential vorticity it is necessary to specify a reference latitude and static stability.

For attribution it is important to describe how $e'_f$ is inverted to find that component of the flow and temperature that can be associated with the PV anomaly. In order to invert $e'_f$, one must have a relationship between $\psi'$ and $\phi'$. Such a relationship is provided by the divergence equation. In enforcing this relationship, approximate versions of the divergence equation which are consistent with the assumption of balanced flow are utilised (e.g. Xu 1994, Davis 1992, Raymond 1992, McIntyre and Norton 1994, Gent and McWilliams 1983). Note however that if the rate of change of divergence is known, there seems to be no reason why the PV cannot be inverted using the unapproximated divergence equation (Bates et al 1995). Assuming that the basic state is geostrophically balanced, the divergence equation can be written in the form:

$$\nabla_h^2 \phi' = \nabla_h^2 (f\psi') + \nabla.(\mathbf{F}_1 + \mathbf{F}_2) \qquad (4)$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ represent those terms which are linear and non-linear in $\psi'$, respectively, and $\nabla_h^2$ represents the horizontal Laplacian operator. Symbolically the inversion of (4) can be written as:

$$\phi' = f\,\psi' + \phi'_a \quad \text{where } \phi'_a = \phi'_1 + \phi'_2 \qquad (5a)$$

and

$$\phi'_1 = \nabla_h^{-2}(\nabla.\mathbf{F}_1) \text{ while } \phi'_2 = \nabla_h^{-2}(\nabla.\mathbf{F}_2) \qquad (5b)$$

In this eq. $\phi'_1$ and $\phi'_2$ represent parts of the geopotential field which have a linear and non-linear dependence on $\psi'$, respectively. The $\nabla_h^{-2}$ symbol represents the inverse horizontal Laplacian operator. Using (5) in (3) and grouping linear and non-linear terms, one finds that

$$e'_f = \frac{1}{N_0^2}\nabla\cdot\left[-\frac{\partial^2\psi'}{\partial x\partial z}\frac{\partial\bar{\phi}}{\partial z}, - \frac{\partial^2\psi'}{\partial y\partial z}\frac{\partial\bar{\phi}}{\partial z} - \frac{\partial^2\bar{\psi}}{\partial y\partial z}\frac{\partial}{\partial z}(f\,\psi' + \phi'_1)\right.,$$

$$\left.\zeta'\frac{\partial\bar{\phi}}{\partial z} + \bar{\zeta}\frac{\partial}{\partial z}(f\,\psi' + \phi'_1)\right] \qquad (6)$$

$$+ \textit{the divergence of non-linear terms in } \psi'.$$

where $\zeta' = \nabla_h^2\psi'$.

The main aim in this paper is to establish a framework in which the field associated with each element of PV charge is an invariant quantity. Since electrostatics is formulated in this way, it is used here as a guide. The property of invariant attribution is attained in electrostatics by insisting that the operator defining total charge is an invariant quantity. The free space Green's functions[1] associated with this operator are themselves invariant and hence make the field attributed to each element of charge an invariant quantity. An important point to recognise is that the total charge is only equal to the free charge in the idealised situation of a charge in free space; i.e. the part of the charge field that can be moved around in a conservative manner is generally not associated with an invariant operator. Thus, to make the notion of action at a distance as clear in PV thinking as it is in electrostatics, we need to define a *total* charge of PV substance, $e'_t$, that is defined by an invariant operator which is equivalent to the *free* charge of PV substance, $e'_f$, in some idealised circumstance. An idealised circumstance which reduces the operator defining $e'_f$ to an invariant occurs if

(a) $\bar{\psi}$ and $\bar{\phi}$ are independent of x and y with $\frac{\partial^2\bar{\phi}}{\partial z^2} = N_0^2$ a constant,

(b) the coriolis parameter is constant; i.e. $f=f_0$,

(c) $\phi'$ and $\psi'$ are in geostrophic balance; i.e. $\phi'_1 = \phi'_2 = 0$,

(d) the non-linear terms are negligible.

[1]By free space Green's function, we mean the Green's function subject to the boundary condition that all derivatives of the state variable diminish to zero at infinity. Also, recall that the operator defining total charge in electrostatics is the three-dimensional Laplacian and that the state variable which is operated on is the electric potential. For the type of PV discussed here the state variable is the streamfunction, $\psi$.

This idealised background state has, in fact, a PV equal to the reference $PV_0$ defined earlier so letting $\bar{\psi}_0$ and $\bar{\phi}_0$ define this state and substituting into (6) gives the definition of total charge density; viz,

$$e'_t = \frac{1}{N_0^2} \nabla . \left[ - \frac{\partial^2 \psi'}{\partial x \partial z} \frac{\partial \bar{\phi}}{\partial z}_0, - \frac{\partial^2 \psi'}{\partial y \partial z} \frac{\partial \bar{\phi}}{\partial z}_0, \zeta' \frac{\partial \bar{\phi}}{\partial z}_0 + f_0^2 \frac{\partial \psi'}{\partial z} \right]$$

$$= \nabla_h^2 \psi' + \frac{f_0^2}{N_0^2} \frac{\partial^2 \psi'}{\partial z^2} \qquad (7a)$$

Remarkably, the total PV charge density for highly accurate forms of the PV is equivalent to the total charge PV density given in BT.[2] The Green's function for the operator defining (7) implies that the part of the streamfunction field, $\delta\psi$, attributable to each element, $e'_t \delta V$, of the PV substance contained within the infinitessimal volume $\delta V$ is

$$\delta\psi = - \frac{1}{4\pi} \left( \frac{e'_t \delta V}{r} \right) \qquad (7b)$$

where $r^2 = (x - x_0)^2 + (y - y_0)^2 + [(N_0/f_0)\ (z - z_0)]^2$ and $(x_0, y_0, z_0)$ is the position of the element. Thus, $e'_t\ \delta V$ gives the total charge in the elemental region $\delta V$ and (7b) gives the field induced by that element of charge. As in electrostatics the difference between the total charge and the free charge is the *bound* charge $e'_b = e'_t - e'_f$. Subtracting (6) from (7a) gives

$$e'_b = e'_{bL} + e'_{bN}, \qquad (8a)$$

where $e'_{bL}$ and $e'_{bN}$ are linear and non-linear functions of $\psi'$, respectively. The algebraic expressions for these quantities are

$$e'_{bL} = \frac{1}{N_0^2} \nabla . \left[ - \frac{\partial^2 \psi'}{\partial x \partial z} \left( \frac{\partial \bar{\phi}}{\partial z}_0 - \frac{\partial \bar{\phi}}{\partial z} \right), - \frac{\partial^2 \psi'}{\partial y \partial z} \left( \frac{\partial \bar{\phi}}{\partial z}_0 - \frac{\partial \bar{\phi}}{\partial z} \right) + \frac{\partial^2 \bar{\psi}}{\partial y \partial z} \frac{\partial}{\partial z} (f\psi' + \phi'_1), \right.$$

$$\left. \zeta' \left( \frac{\partial \bar{\phi}}{\partial z}_0 - \frac{\partial \bar{\phi}}{\partial z} \right) + [f_0^2/f - \bar{\zeta}]\ f \frac{\partial \psi'}{\partial z} - \bar{\zeta} \frac{\partial \phi'}{\partial z}_1 \right] \qquad (8b)$$

and

[2] In fact the quasi-geostrophic form has the complication of including the air density within the vertical derivatives. The Ertel-Rossby PV substance has no such complication.

$$e'_{bN} = -\frac{1}{N_0^2}\nabla.\left[-\frac{\partial^2\psi'}{\partial x\partial z}\,f\,\frac{\partial\psi'}{\partial z},\ -\frac{\partial^2\psi'}{\partial y\partial z}\,f\,\frac{\partial\psi'}{\partial z},\ \zeta' f\,\frac{\partial\psi'}{\partial z}\right]$$

$$-\frac{1}{N_0^2}\nabla.\left[0,\ -\frac{\partial^2\bar{\psi}}{\partial y\partial z}\,\frac{\partial\phi'}{\partial z}2,\ \bar{\zeta}\,\frac{\partial\phi'}{\partial z}2\right]$$

$$-\frac{1}{N_0^2}\nabla.\left[-\frac{\partial^2\psi'}{\partial x\partial z}\,\frac{\partial\phi'}{\partial z}a,\ -\frac{\partial^2\psi'}{\partial y\partial z}\,\frac{\partial\phi'}{\partial z}a,\ \zeta'\,\frac{\partial\phi'}{\partial z}a\right] \qquad (8c)$$

At this point, note that the vectors defined by the square brackets in (7a) and (8b,c) are, respectively, the counterparts of the electric field and polarisation vectors used in electrostatics. In electrostatics these vectors can always be defined in terms of some combination of the components of the gradient of electric potential. The above expressions give the PV counterparts of electric field and displacement in terms of some combination of the components of the vorticity vector and the perturbation potential temperature and *not* in terms of the components of the gradient of the state variable $\psi'$. Given the importance of the vorticity vector in potential vorticity, this is not surprising. However, the divergence of the vorticity vector is zero and it turns out that (7) and (8) can be written in terms of the divergence of vectors whose components are made up of the components of the gradient of the streamfunction and the vertical derivitive of $\phi'_a$. The alternative expression for the total charge is

$$e'_t = \nabla.\left(\frac{\partial\psi'}{\partial x},\ \frac{\partial\psi'}{\partial y},\ \frac{f_0^2}{N_0^2}\frac{\partial\psi'}{\partial z}\right) \qquad (9a)$$

Here, the term in brackets gives the counterpart of the electric field vector.

For the bound charge, (8c) already gives the non-linear part of the bound charge in terms of products involving first order derivatives of $\psi'$, $\phi'_2$ and $\phi'_a$. The linear part is also expressible in such a form; *viz*,

$$e'_{bL} = \frac{1}{N_0^2}\nabla.\left[(N_0^2 - \bar{N}^2)\,\frac{\partial\psi'}{\partial x},\ (N_0^2 - \bar{N}^2)\,\frac{\partial\psi'}{\partial y} + \frac{\partial^2\bar{\psi}}{\partial y\partial z}\,\frac{\partial}{\partial z}(f\psi' + \phi'_1),\right.$$

$$\left.[f_0^2/f - \bar{\zeta}]\,f\,\frac{\partial\psi'}{\partial z} + \frac{\partial^2\bar{\phi}}{\partial y\partial z}\,\frac{\partial\psi'}{\partial y} - \bar{\zeta}\,\frac{\partial\phi'}{\partial z}1\right] \qquad (9b)$$

where $\bar{N}^2 = \frac{\partial^2 \bar{\phi}}{\partial z^2}$ is the static stability of the background state. Here, the term in the square brackets gives the negative of the counterpart to the polarisation vector. Note that provided $N_0$ is less than or equal to $\bar{N}$ the horizontal part of the polarisation vector will tend (provided $\frac{\partial^2 \bar{\psi}}{\partial y \partial z}$ is small) to point in the same direction as the electric field vector. Thus, provided $N_0$ is chosen in this way the bound charge at a point will tend to be of the opposite sign to both the total and free charge. Since the free charge is just the difference between the total and bound charge, it is also expressible in terms of first order derivatives of $\psi'$, $\phi_2'$ and $\phi_a'$. Its full form is rather cumbersome to write down but in the special case of geostrophic balance on an f-plane, it takes on a simpler form which will be explored in the next section.

## 3. Geostrophic balance on an f-plane

Here a summary of the formulation is given making the assumption, for simplicity, that the flow associated with the PV anomaly is in geostrophic balance on an f-plane. In this case the free PV charge becomes:

$$e_f' = e_{fL}' + e_{fN}' \tag{10a}$$

where

$$e_{fL}' = \frac{1}{N_0^2} \nabla . \left[ \bar{N}^2 \frac{\partial \psi'}{\partial x}, \bar{N}^2 \frac{\partial \psi'}{\partial y} - f_0 \frac{\partial^2 \bar{\psi}}{\partial y \partial z} \frac{\partial \psi'}{\partial z}, \bar{\zeta} f_0 \frac{\partial \psi'}{\partial z} - f_0 \frac{\partial^2 \bar{\psi}}{\partial y \partial z} \frac{\partial \psi'}{\partial y} \right] \tag{10b}$$

and

$$e_{fN}' = \frac{f_0}{N_0^2} \nabla . \left[ - \frac{\partial^2 \psi'}{\partial x \partial z} \frac{\partial \psi'}{\partial z}, - \frac{\partial^2 \psi'}{\partial y \partial z} \frac{\partial \psi'}{\partial z}, \zeta' \frac{\partial \psi'}{\partial z} \right]. \tag{10c}$$

The terms in square brackets define the displacement vector for geostrophically balanced PV fields. From these vectors, we may write down the dielectric tensor for PV fields, i.e. $e_f' = \nabla . \underline{D}'$, where the vector $\underline{D}'$ is defined as:

$$\underline{D}' = ( \underset{\sim}{\varepsilon}_L + \underset{\sim}{\varepsilon}_N ) \nabla \psi' \tag{11}$$

where $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}_L + \underset{\sim}{\varepsilon}_N$ is the dielectric tensor with components $\varepsilon_{ij}$ with i and j ranging from 1 to 3. The linear and non-linear components have been separated viz: $e_{fL}' = \nabla.( \underset{\sim}{\varepsilon}_L \nabla \psi')$ and $e_{fN}' = \nabla.( \underset{\sim}{\varepsilon}_N \nabla \psi')$. From

inspection of equation (10) the components of the linear part of the dielectric tensor are:

diagonal components $\varepsilon_{11} = \varepsilon_{22} = \dfrac{\bar{N}^2}{N_0^2} \qquad \varepsilon_{33} = \dfrac{f_0 \bar{\zeta}}{N_0^2}$ (12a)

off-diagonal components: $\varepsilon_{23} = \varepsilon_{32} = \dfrac{f_0 \bar{u}_z}{N_0^2}$ & other comps. zero

The only non-zero part of the non-linear component, $\varepsilon_{13}$, is proportional to the relative vorticity vector of the anomaly i.e.

$$\varepsilon_{13} = \frac{f_0}{N_0^2} \left( - \frac{\partial^2 \psi'}{\partial x \partial z}, - \frac{\partial^2 \psi'}{\partial y \partial z}, \nabla_h^2 \psi' \right) \qquad (12b)$$

It can be seen that the linear part only depends on the background static stability and vorticity. On the other hand the non-linear part depends on the geopotential itself via the relative vorticity; this is not known until the inversion solution has been obtained.

The definition of the bound PV, as given in BT, is:

$$e_b' = - \nabla . [ (\underset{\sim}{\varepsilon} - 1) \nabla \psi' ] \qquad (13)$$

In the quasi-geostrophic case, as described by BT, the bound PV was due to the equivalent of a vertical polarization (as $\underset{\sim}{\varepsilon} - 1$ is only non-zero in the vertical component; see section 3 (i) below). It can be seen that in the Ertel-Rossby case it is due to a linear contribution with both horizontal and vertical polarization and a new non-linear contribution associated with $e'_{fN}$. Therefore, the non-linear parts of the PV are, in this electrostatics analogy, components of the bound charge. This provides an attractive physical picture of the differences between the fields attributable to Ertel-Rossby and quasi-geostrophic forms of the PV.

The field induced by the non-linear part of the bound charge will now be shown to rapidly diminish with distance away from an isolated free PV charge. The net non-linear bound charge in a volume, V, is equal to the volume integral of $e'_{bN}$. By Gauss's theorem, this integral is equal to the flux of the polarisation field through the

surface containing the volume V. Suppose that the amplitude of $\psi'$ was inversely proportional to the distance r away from a localised free PV anomaly. Then, using the fact that $e'_{bN} = - e'_{fN}$ (as described in section 3) and using equation (10b), the non-linear part of the polarisation field would be proportional to $1/r^5$. Consequently, the bound charge contained in a sphere of radius r enclosing the anomaly would be proportional to $1/r^3$. Thus, the total bound charge associated with this isolated free PV anomaly is zero and hence its effect on the far-field is zero. Note also that, in this circumstance the amplitude of the bound charge would be proportional $1/r^6$. This qualitative discussion suggests that the effect of the non-linear part of the bound charge on the streamfunction field is localised near isolated anomalies and may not affect the far-field features of the streamfunction field.

Note that the above argument applies equally well to the less approximated form of the non-linear part of the bound charge given in (8b). In Section 5, we illustrate this localisation of non-linear bound charge with an example from semi-geostrophic theory.

## 4. Comparison with the quasi-geostrophic PV

*(i) Dielectric tensor*

A simplification to the general theory described in the previous section is provided by assuming a background PV which is only a function of height. Then there is a background potential vorticity $\overline{PV} = f \bar{N}^2 \theta_0 / gp$, where $\bar{N}$ is the background Brunt-Vaisala frequency which is taken as a function of height only. The only changes to the equations described in the previous section are that: $\bar{N}^2 = \bar{N}^2(z)$, $\bar{\zeta} = f$, and $\bar{\psi}_{yz} = 0$. Therefore the dielectric tensor components become:

$$\text{linear part:} \quad \varepsilon_{11} = \varepsilon_{22} = \frac{\bar{N}^2}{N_0^2} \qquad \varepsilon_{33} = 1 \quad \& \quad \text{other comps. zero} \tag{14}$$

The non-linear components are unchanged.

The linear part only depends on the background static stability

which is a known function of height. The non-linear part depends on the geopotential itself via the relative vorticity; this is not known until the inversion solution has been obtained.

Compare this dielectric tensor for the Ertel-Rossby PV with that obtained in BT for the quasi-geostrophic PV. Using the same coordinates as above the relevant quasi-geostrophic forms are: $e' = \bar{\nabla} \cdot \underline{D}'_Q$ where $\underline{D}'_Q = \underset{\sim}{\varepsilon}\, \bar{\nabla} \psi'$ and the quasi-geostrophic dielectric tensor only has a linear part with components given by:

$$\varepsilon_{11} = \varepsilon_{22} = 1 \qquad \varepsilon_{33} = \frac{N_0^2}{\bar{N}^2} \qquad \& \quad \text{other comps. zero} \qquad (15)$$

It is clear by comparing the expressions in equations (14) and (15) for the dielectric constant that the quasi-geostrophic PV is <u>not</u>, in general, equal to the linear part of the Ertel-Rossby PV. They only become equal in the case when there is no vertical variation in the background static stability so that we may define $\bar{N}^2 = N_0^2$ and equations (14) and (15) are then identical. The implications of this difference are considered in the next section for the problem of a point PV anomaly residing beneath the tropopause; the solution in the quasi-geostrophic case having been given in BT.

*(ii) The role of the tropopause*

An interesting case arises when there is a tropopause like jump in the static stability above a point anomaly, Q, of perturbation free charge density. To simplify the analysis, assume that the static stability of the troposphere is equal to the reference static stability and that Q is of small enough amplitude and is sufficiently distant from the tropopause that in the vicinity of the tropopause the effect of the non-linear bound charge in the troposphere is zero. In this circumstance, it is as if the total charge distribution in the troposphere was precisely equal to the free charge distribution. In order that there be no free charge at the tropopause the vertical component of the displacement vector defined in (10b) must be continuous across the tropopause. With $\psi'$ continuous across the tropopause (and hence $\phi'$), this condition is met when the potential temperature, $\psi'_z$, is continuous across the boundary. For a point

charge a height $a$ below the tropopause, these conditions are satisfied if

$$\psi'_t = -\frac{Q}{4\pi}\left( \frac{1}{(x^2 + y^2 + z^2\ N_t^2/f^2)^{1/2}} + \frac{n}{(x^2 + y^2 + (z - 2a)^2\ N_t^2/f^2)^{1/2}} \right)$$

$$\psi'_s = -\frac{Q\ \ (1 + n\ )}{4\pi\ (x^2 + y^2 + (z - \ell\ a\ )^2\ N_s^2/f^2\ )^{1/2}} \qquad (16)$$

where $n = -\ell\ /\ m$ where $\ell = (1 - N_t/N_s)$ and $m = (1 + N_t/N_s)$. The coordinates (x, y, z) indicate the distance from the point charge. The subscripts t (for troposphere) and $s$ (for stratosphere) indicate that the form of the streamfunction indicated applies below and above $z = a$, respectively.

The streamfunction form described by (16) is identical to that for a free charge of QG PV given in BT except here $n$ has the opposite sign. This is a direct consequence of the different form of the dielectric constant for $e'_{fL}$ compared with that for QG PV. Note also that the boundary condition at the tropopause imposed in the QG case, to give zero $e'_{fL}$ at the interface, is different (continuity of $\psi'_z/N^2$); this is also a consequence of the difference in the dielectric constant. It is worth noting that the QG boundary condition implies an unphysical jump in potential temperature at the tropopause. This structure is removed in the Ertel-Rossby case.

One of the important features of the above solution is that we expect it to be the solution for an element of free charge at large distances from the anomaly due to the rapid diminution of the effects of the non-linear terms at large distance. For example for $z \to \infty$ the limit is $\psi'_s \to -\ Q\ (\ 1\ -\ \ell\ /\ m\ )\ /\ 4\ \pi\ r$ whereas in the quasi-geostrophic case discussed in BT the equivalent limit is given by $\psi'_s \to -\ Q\ (\ 1\ +\ \ell\ /\ m\ )\ /\ 4\ \pi\ r$. This shows that the bound charge associated with the change in static stability at the tropopause acts in an incorrect way in the QG system tending to <u>overestimate</u> the size of the geopotential in the far-field.

## 5. Inversion solutions for an Ertel-Rossby PV anomaly

In this section the case of a single PV anomaly embedded in a uniform background PV with no flow is considered as it provides analytical solutions which make the previous theoretical formulation explicit. For this situation we can take $\bar{N}^2 = N_0^2$, $\overline{PV} = \overline{PV}_0$, and $\rho = \rho_0$ as constants. In this case the linear part of the bound charge density $e'_{bL} = 0$ (see equation 9b), $e'_{bN} = -e'_{fN}$, and $e'_{fL} = e'_t$. Here we exploit an important mathematical simplification that arises if the PV anomaly is spherically symmetric in the radial coordinate, r, defined following equation (7b). The geostrophic flow assumption will be made by setting $\psi' = \phi'/f_0$.

(i) *Quasi-geostrophic case*

The quasi-geostrophic PV, q′, can be expressed, using the radial coordinate transformation, as:

$$q' = \frac{1}{f_0 r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi'}{\partial r} \right) \qquad (17)$$

This, as has been noted earlier, shows that the quasi-geostrophic PV is, in this case, equal to the linear part of the Ertel-Rossby PV, $e'_{fL}$. The magnitude of the anomalies are equal if $q' = e'_{fL} = (PV - PV_0)\ f_0 \ / \ PV_0$

A simple example of such a PV anomaly is a "ball" which has a constant but different PV compared to the rest of the atmosphere:

$$\text{Ball solution:} \quad r > b \qquad q' = 0 \qquad \Rightarrow \phi' = -\frac{f_0}{3} \tilde{q} \frac{b^3}{r}$$

$$r < b \qquad q' = \tilde{q} \qquad \Rightarrow \phi' = -f_0 \tilde{q} \left( \frac{b^2}{2} - \frac{r^2}{6} \right) \qquad (18)$$

Such a ball anomaly of PV represents, in some senses, an ideal shaped anomaly in that it has equal dimensions in all three directions as long as the normal scaling of the vertical coordinate (N/f) is made as befits the usual shallow atmosphere approximation. As mentioned by HMR it is therefore illuminating to note that the vorticity anomaly is related to the static stability anomaly in the

following way:

$$r < b \qquad \frac{\zeta'}{f_0} = 2\,\frac{\partial\theta'/\partial z}{\partial\bar{\theta}/\partial z} = \frac{2\,\tilde{q}}{3\,f_0} \qquad (19)$$

$$r > b \qquad \frac{\zeta'}{f_0} = -\,\frac{\partial\theta'/\partial z}{\partial\bar{\theta}/\partial z}$$

Hence, as noted in HMR, the vorticity anomaly inside the PV anomaly is of twice the magnitude of the static stability anomaly irrespective of the PV magnitude or of the anomaly dimension b. This partition will change, however, as the anomaly changes shape. As Hoskins et al 1985 note, as the PV anomaly becomes tall and narrow a greater proportion is partitioned into vorticity whereas for a wide and shallow anomaly a greater proportion is in the static stability.

(ii) *Semi-geostrophic PV*

An alternative estimate of the Ertel-Rossby PV is given by the so-called semi-geostrophic potential vorticity, SPV, defined as:

$$SPV = \frac{1}{\rho_0}\,\underline{\zeta}_{SG} \cdot \nabla\theta = \frac{1}{\rho_0}\,\nabla \cdot (\theta\,\underline{\zeta}_{SG}) \qquad (20)$$

$$\text{where} \quad \underline{\zeta}_{SG} = \underline{\zeta}_{G} + \frac{1}{f_0}\,\underline{J}_{xyz}(u, v)$$

Note that here suffices x, y, and z refer to spatial derivatives.

The semi-geostrophic vorticity has additional Jacobian terms which do not appear in a simple geostrophic estimation of vorticity. These are formally small in situations where the semi-geostrophic approximation is valid. However, following Shutts and Cullen 1987, it is possible to make progress whilst retaining the small quadratic terms and remaining in Cartesian coordinates. It is possible, as described in the Appendix, to obtain an expression for the semi-geostrophic free PV charge, $s'_f = (SPV - \overline{SPV})\,f_0/\,\overline{SPV}$, using the radial coordinate r:

$$s'_f = \frac{1}{f_0 r^2}\,\frac{\partial}{\partial r}\left( r^2\,\frac{\partial\phi'}{\partial r} + \frac{1}{f_0^2}\,r\,\left(\frac{\partial\phi'}{\partial r}\right)^2 + \frac{1}{3f_0^4}\left(\frac{\partial\phi'}{\partial r}\right)^3 \right) \qquad (21)$$

Notice that the first term is linear and assumes the mathematical form of the quasi-geostrophic PV first given in equation (17), the second term is quadratic, and the third term is cubic in form.

It can be shown that the solution of equation (21) for a localized ball of SPV with $s'_f = \tilde{s}$ for $r < b$ and $s' = 0$ for $r > b$ is given by the following:

$$\frac{\partial \phi'}{\partial r} = f_0^2 \, r \left( \left( 1 + \frac{\tilde{s}}{f_0} \frac{b^3}{r^3} \right)^{1/3} - 1 \right) \qquad \text{for } r > b$$

$$\frac{\partial \phi'}{\partial r} = f_0^2 \, r \left( \left( 1 + \frac{\tilde{s}}{f_0} \right)^{1/3} - 1 \right) \qquad \text{for } r < b \tag{22}$$

These solutions are related to those first given by Shutts 1991 for the total flow including a point anomaly. Here though the strength of the flow is explicitly given in terms of the strength of the PV anomaly. The solutions are in terms of the radial gradient of the potential as for $r > b$ the solution is in the form of an indefinite integral with no closed form.

Inside the ball anomaly it is possible to integrate equation (22) to obtain the potential itself:

$$\Rightarrow \quad \phi' = f_0^2 \frac{r^2}{2} \left( \left( 1 + \frac{\tilde{s}}{f_0} \right)^{1/3} - 1 \right) + \phi'_{00} \qquad \text{for } r < b$$

$$\text{where } \phi'_{00} = - f_0^2 \left( \int_b^{\infty} r \left( \left( 1 + \frac{\tilde{s}}{f_0} \frac{b^3}{r^3} \right)^{1/3} - 1 \right) dr \right.$$

$$\left. + \frac{b^2}{2} \left( \left( 1 + \frac{\tilde{s}}{f_0} \right)^{1/3} - 1 \right) \right) \tag{23}$$

For $r > b$ outside the anomaly $\phi'$ can only be expressed as an integral but for large radii we can show that:

$$\text{For } r \to \infty \qquad \frac{\partial \phi'}{\partial r} \approx \frac{f_0^2}{3} \frac{\tilde{s}}{f_0} \frac{b^3}{r^2} \left( 1 - \frac{1}{3} \frac{\tilde{s}}{f_0} \frac{b^3}{r^3} + \ldots \right)$$

(24)

$$\Rightarrow \phi' \approx - \frac{f_0^2}{3} \frac{\tilde{s}}{f_0} \frac{b^3}{r} \left( 1 - \frac{1}{12} \frac{\tilde{s}}{f_0} \frac{b^3}{r^3} + \ldots \right)$$

So, by comparing equations (24) and (18), the semi-geostrophic solution, for large radius, has the same radial dependence as the quasi-geostrophic one.

In figure 1 the semi-geostrophic and quasi-geostrophic solutions are compared for the case when $s'_f = q' = 4\ f_0$. This strong positive anomaly exhibits the important feature that the quasi-geostrophic solution involves local regions of static instability and so is outside its domain of validity. In contrast the semi-geostrophic solution does not exhibit static instability for any magnitude anomaly. It can be seen that inside the anomaly the QG solution has much larger static stability and vorticity than its semi-geostrophic counterpart. This is associated with the fact that the QGPV is additive in terms of vorticity and stability whereas the SPV is multiplicative. From equation (22) it is clear that the semi-geostrophic solution inside the anomaly exhibits exactly the same relationship between the (geostrophic) vorticity and static stability anomalies as the quasi-geostrophic case of equation (19). The connection of these anomalies to the magnitude of the PV itself can also be obtained from equation (22):

$$r < b \qquad \frac{\zeta'}{f_0} = 2 \frac{\partial\theta'/\partial z}{\partial\bar{\theta}/\partial z} = 2 \left( \left( 1 + \frac{\tilde{s}}{f_0} \right)^{1/3} - 1 \right)$$

An approximation to this relationship in the region outside the PV anomaly can be obtained by ignoring the horizontal vorticity components (and those associated with the Jacobian term):

$$r > b \qquad \frac{\zeta'}{f_0} = - \frac{\frac{\partial\theta'}{\partial z} / \frac{\partial\bar{\theta}}{\partial z}}{1 + \frac{\partial\theta'}{\partial z} / \frac{\partial\bar{\theta}}{\partial z}}$$

In contrast to the QG solution this latter relationship implies that anticyclonic vorticity is limited such that the absolute vorticity is positive.

(iv) *Semi-geostrophic dielectric constant*

Using the terminology of section 2 equation (21) can be written in the following form:

$$s'_f = \nabla \cdot \underline{D}'_s \qquad (25)$$

where the vector field associated with a semi-geostrophic PV anomaly written in the spherical coordinates is $\underline{D}'_s = \epsilon_s \frac{1}{f_0} \frac{\partial \phi'}{\partial r} \hat{\underline{r}}$ where $\hat{\underline{r}}$ is a unit vector in the radial direction. The quantity $\epsilon_s$ is the semi-geostrophic dielectric constant and has the following form obtained by comparing equations (21) and (25):

$$\epsilon_s = 1 + \frac{1}{f_0^2 r} \frac{\partial \phi'}{\partial r} + \frac{1}{3} \left( \frac{1}{f_0^2 r} \frac{\partial \phi'}{\partial r} \right)^2 \qquad (26)$$

It is clear that the SG dielectric constant is not unity even in the case being considered here in which the background static stability is uniform i.e. as shown in section 3 there is a non-linear contribution to the bound charge. To complete the electrostatics analogy for semi-geostrophic theory then the equivalent of the electric and polarization fields can be defined as follows:

$$s'_f + s'_b = \nabla \cdot \underline{E}'_s \qquad \text{and} \qquad \underline{E}'_s = \frac{1}{f_0} \frac{\partial \phi'}{\partial r} \hat{\underline{r}}$$

$$s'_b = - \nabla \cdot \underline{P}'_s \qquad \text{and} \qquad \underline{P}'_s = \chi_s \frac{1}{f_0} \frac{\partial \phi'}{\partial r} \hat{\underline{r}}$$

where $\chi_s = \epsilon_s - 1$ and $s'_b$ is the semi-geostrophic bound charge.

Using the analytical solution of equation (22) allows us to write an explicit expression for the SG dielectric constant for the ball charge:

$$\epsilon_s = 1 + \frac{1}{3} \left( \left( 1 + \frac{\tilde{s}}{f_0} \right)^{1/3} - 1 \right) \left( \left( 1 + \frac{\tilde{s}}{f_0} \right)^{1/3} + 2 \right) \quad r < b$$

$$\epsilon_s = 1 + \frac{1}{3} \left( \left( 1 + \frac{\tilde{s}}{f_0} \frac{b^3}{r^3} \right)^{1/3} - 1 \right) \left( \left( 1 + \frac{\tilde{s}}{f_0} \frac{b^3}{r^3} \right)^{1/3} + 2 \right) \quad r > b$$

In figure 2(a) the dependence of $\epsilon_s$ on $\tilde{s}$ for $r < b$ is given. It is interesting to note that the susceptibility, $\chi_s$, has the same sign as the normalised PV anomaly, $\tilde{s}$. In figure 2(b) the radial dependence of $\epsilon_s$ is given fixing $\tilde{s} = 3\ f_0$. It can be seen that $\epsilon_s$ rapidly approaches unity away from the PV anomaly.

The bound charge density, $s'_b$, can also be evaluated from the ball solution:

$$s'_b = -f\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3}-1\right)^2\left(\left(1+\frac{\tilde{s}}{f_0}\right)^{1/3}+2\right) \quad \text{for } r < b$$

$$s'_b = 3\ f_0\left\{\frac{\left(1+\frac{2}{3}\frac{\tilde{s}}{f_0}\frac{b^3}{r^3}\right)}{\left(1+\frac{\tilde{s}}{f_0}\frac{b^3}{r^3}\right)^{2/3}}-1\right\} \quad \text{for } r > b \qquad (27)$$

It can be seen that the sign of $s'_b$ is independent of the sign of the PV anomaly. Within the ball $s'_b$ is negative. Outside the ball it is positive and at large radii it has a $1/r^6$ variation. Note that the dependence of $s'_b$ on the radius is consistent with the heuristic argument given at the end of Section 3. Using Gauss's integral theorem, it is straightforward to prove that the volume integral of the bound charge density, $s'_b$, is zero.

Figure 3 shows the form of the bound charge $s'_b$ as a function of radius for a range of values of $\tilde{s}$. In figure 3(a) the radial dependence of $s'_b\ /\ \tilde{s}$ is given for $\tilde{s}\ /\ f_0 = 3$. This bound charge, $s'_b$, is therefore composed of a central negative PV anomaly surrounded by an equal amount of positive PV.

Figure 3(b) gives the dependence of $s'_b$ on the SPV anomaly of strength, $\tilde{s}$, in the region $r < b$ where it is a constant. It can be seen that inside the ball charge $s'_b$ is always negative irrespective of the sign of the SPV anomaly. Furthermore for the most extreme anticyclonic anomaly allowed, $\tilde{s}\ /\ f_0 = -1$ giving a total PV, anomaly plus background, inside the anomaly of zero, the bound charge contribution $s'_b\ /\ f_0 = -2$ so that the total anomaly, free and bound

charge, is $(s'_f + s'_b) / f_0 = -3$. Outside the SPV anomaly the bound charge is positive.

It is illuminating to consider the differences between the QG and SG solutions. The differences in flow and temperature are given in figure 4 for the case described by figure 1. It can be seen that the difference is essentially similar to that due to a negative anomaly located within the semi-geostrophic ball charge as can be seen by the anticyclonic flow and temperature dipole. This difference reflects the high concentration of negative bound charge within the SG ball.

As, in this case, the semi-geostrophic geopotential tends to the quasi-geostrophic form at large radii we have noted that an equivalent QG anomaly can be constructed via $q' = s'_f$. It is then clear that the reciprocal of the semi-geostrophic dielectric constant gives a measure of the modification to the QG field made in using the semi-geostrophic equations. From figure 2(b) it is clear that this modification is only large close to the anomaly.

## 6. Discussion

In this paper we have shown that the electrostatics analogy applies to both the Ertel-Rossby potential vorticity and the quasi-geostrophic approximation to potential vorticity. For a PV anomaly embedded in an arbitrary zonally-averaged flow, with background PV varying with latitude and height, the analogy consists of describing the atmosphere in terms of a non-linear dielectric medium. The associated dielectric "constant" is composed of a linear and a non-linear part. The linear part is similar but not exactly the same as in the quasi-geostrophic case. The difference exists if there is a vertical variation in the background PV such as occurs at the tropopause or if the zonal flow is sheared in the horizontal or vertical. In contrast, in the near neighbouhood of an Ertel-Rossby PV anomaly the dielectric "constant" has a significant non-linear

contribution which produces streamfunction features markedly different to that associated with QG anomalies.

The non-linear contributions are interpreted here as being associated with bound charge. The conceptual simplicity of an element of total PV charge inducing simple fields, such as that inversely dependant on distance from the anomaly, is then retained. The bound PV charge is then caused by the "polarization" of the medium by the field due to the total charge. The non-linear bound charge contribution acts to modify the simple field only locally to the PV anomaly. In the far-field the non-linear bound charge contribution is negligible. This is why many of the essential characteristics associated with a PV anomaly are unchanged from the QG solution e.g. a positive anomaly has cyclonic circulation with warm air above and cold air below as given in figure 1 of BT.

Clearly the simple linear superposition principle that applies in the quasi-geostrophic case does not carry over to the Ertel-Rossby case. However the non-linearity of the dielectric constant is significant only in very close proximity to the anomaly. This means that apart from anomalies which are extremely close together linear superposition will be a rather accurate approximation. Of course one is superposing somewhat different flow structures compared to that given by the quasi-geostrophic case so the solution will be mainly different for that reason rather than from the approximation implied by the linear superposition.

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We would like to thank David Stephenson for pointing out the connection to quantum field theory. Craig Bishop was funded from NERC research grant no. GR3/6920 during this research.

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Figure Captions:

Figure 1: (a) Vertical section showing potential temperature (contour interval 4 K) and normal velocity (contour interval 4 $ms^{-1}$) for a semi-geostrophic PV anomaly of magnitude $\tilde{s}/f_0 = 4$ with a spherical shape in the stretched coordinate. Axes are marked with distances divided by the arbitrary anomaly radius.

(b) As in (a) for a quasi-geostrophic anomaly of the same magnitude.

Figure 2: (a) The semi-geostrophic dielectric constant, $\epsilon_s$, within the potential vorticity anomaly as a function of the anomaly strength $\tilde{s}/f_0$.

(b) Radial variation of the semi-geostrophic dielectric constant for an anomaly of magnitude $\tilde{s}/f_0 = 3$. Radius is normalised by the anomaly radius.

Figure 3: (a) Radial variation of the bound charge for an anomaly of strength $\tilde{s}/f_0 = 3$. The bound charge has been normalised by the anomaly magnitude.

(b) The dependence of bound charge, within the PV anomaly, on the anomaly strength.

Figure 4: The difference in potential temperature and normal velocity between the semi-geostrophic and quasi-geostrophic solutions given in figure 1.

## APPENDIX

Following Shutts and Cullen 1987 one can introduce a new potential function, $P = \phi + \frac{f_0^2}{2}(x^2 + y^2)$, which has the property that $\underline{\zeta}_{SG} = \frac{1}{f_0^3} \nabla P_x \times \nabla P_y$. Then the SPV, as defined in equation (20), can be written as the determinant of a matrix:

$$SPV = \frac{\theta_0}{\rho_0 g f_0^3} \det \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}$$

Using a decomposition into a background mean state, denoted by an overbar, with uniform potential vorticity and a local deviation, denoted by a prime, then

$$P = \bar{P} + \phi' \quad \text{where } \bar{P} = \bar{\phi} + \frac{f_0^2}{2}(x^2 + y^2)$$

Taking $\bar{\phi} = \frac{N_0^2 z^2}{2} + g z$ gives $\overline{SPV} = \frac{\theta_0}{\rho g f^3} \bar{P}_{xx} \bar{P}_{yy} \bar{P}_{zz} = \frac{\theta_0 f_0 N_0^2}{\rho g}$ and if the anomaly of semi-geostrophic PV is defined as before as $s_f' = (SPV - \overline{SPV})\, f_0 / \overline{SPV}$ then the following can be derived:

$$s_f' = \frac{1}{f_0}\left( \phi'_{xx} + \phi'_{yy} - \frac{f_0^2}{N_0^2} \phi'_{zz} \right)$$

$$+ \frac{1}{f_0 N_0^2}\left( (\phi'_{xx} + \phi'_{yy})\, \phi'_{zz} - \phi'^2_{xz} - \phi'^2_{yz} \right)$$

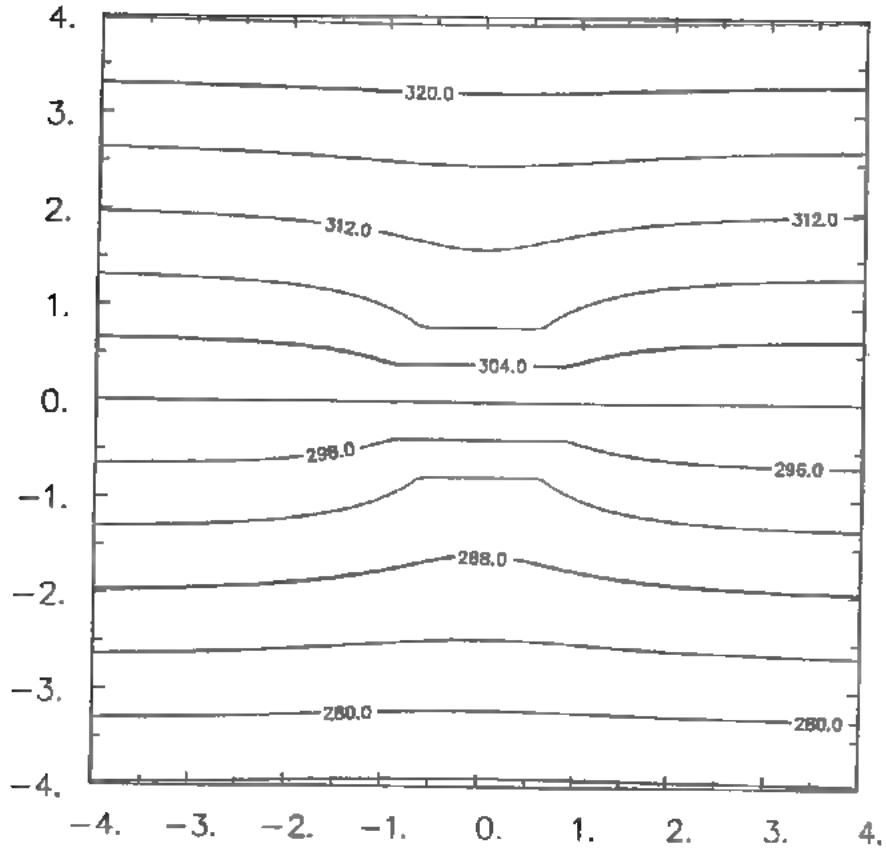
$$+ \frac{1}{f_0^3}\left( \phi'_{xx} \phi'_{yy} - \phi'^2_{xy} \right)$$

$$+ \frac{1}{f_0^3 N_0^2} \det \begin{bmatrix} \phi'_{xx} & \phi'_{xy} & \phi'_{xz} \\ \phi'_{yx} & \phi'_{yy} & \phi'_{yz} \\ \phi'_{zx} & \phi'_{zy} & \phi'_{zz} \end{bmatrix}$$

Hence $s_f'$ is the sum of the linear QG term, a quadratic term (that also appears in a geostrophic estimate of the PV), and one quadratic and one cubic term specifically associated with the SPV. In

spherical polar coordinates, using r as defined in section 2, $s'_f$ can be written with some algebraic manipulation as the sum of the three terms given in equation (21).

It is interesting to note that the SPV has a spherically symmetric form for a spherical PV anomaly field using the radial coordinate. (Note that the component terms are not all individually symmetrical. Also the mean state which has been subtracted from the geopotential is, apart from constant geopotential and potential temperature factors, also spherically symmetric in r; i.e. $\bar{P} = \phi_0 + g\,z + \frac{f_0^2}{2} r^2$.) It is clear therefore that the Jacobian terms for the semi-geostrophic vorticity, although of small magnitude, are sufficient to make the Ertel-Rossby PV have a symmetric form.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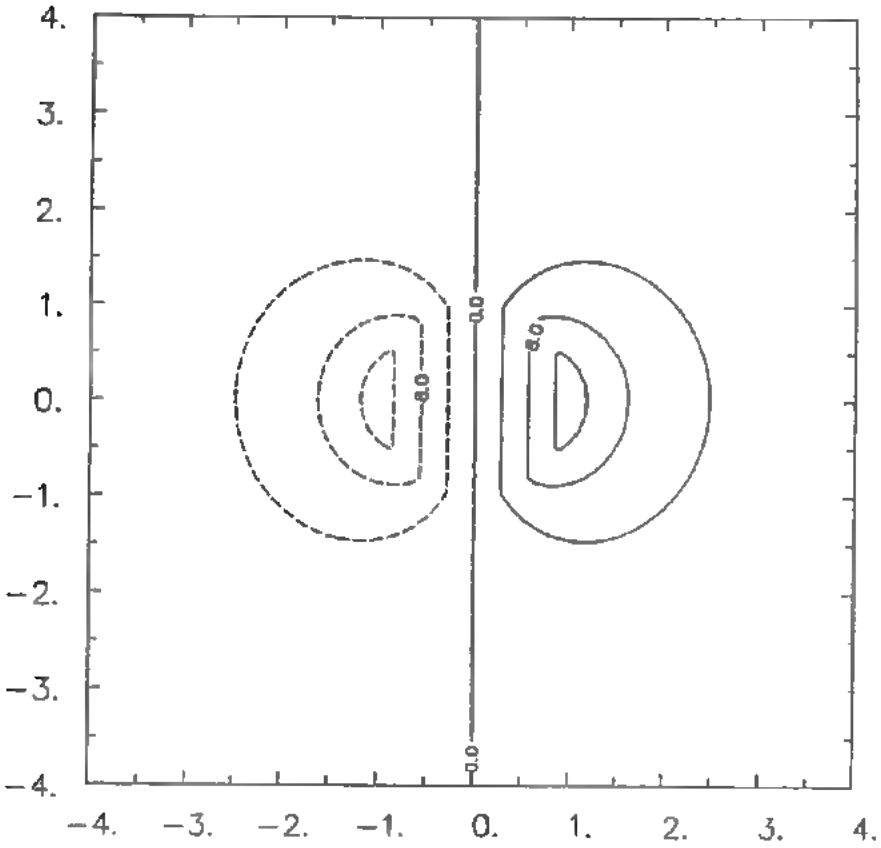
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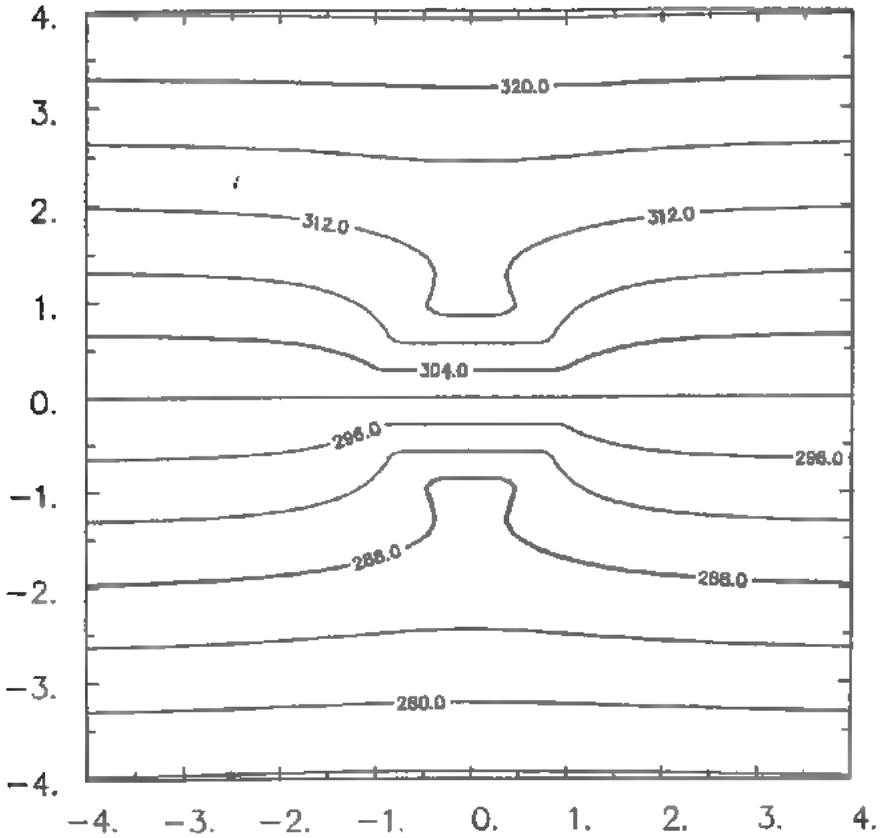
FIG 1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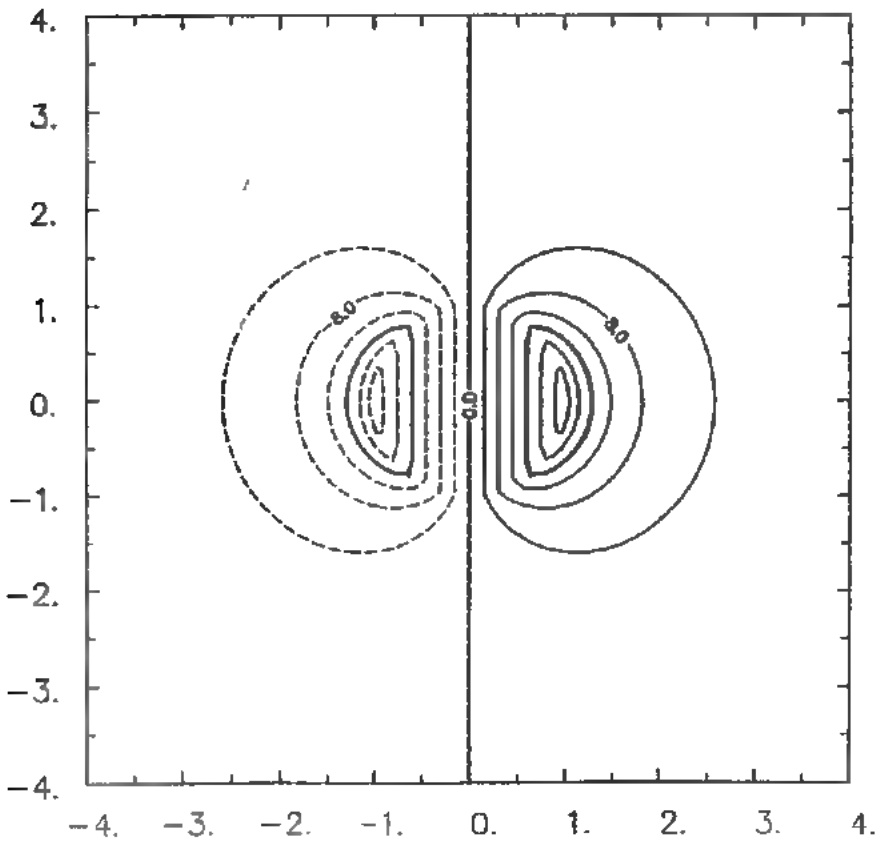
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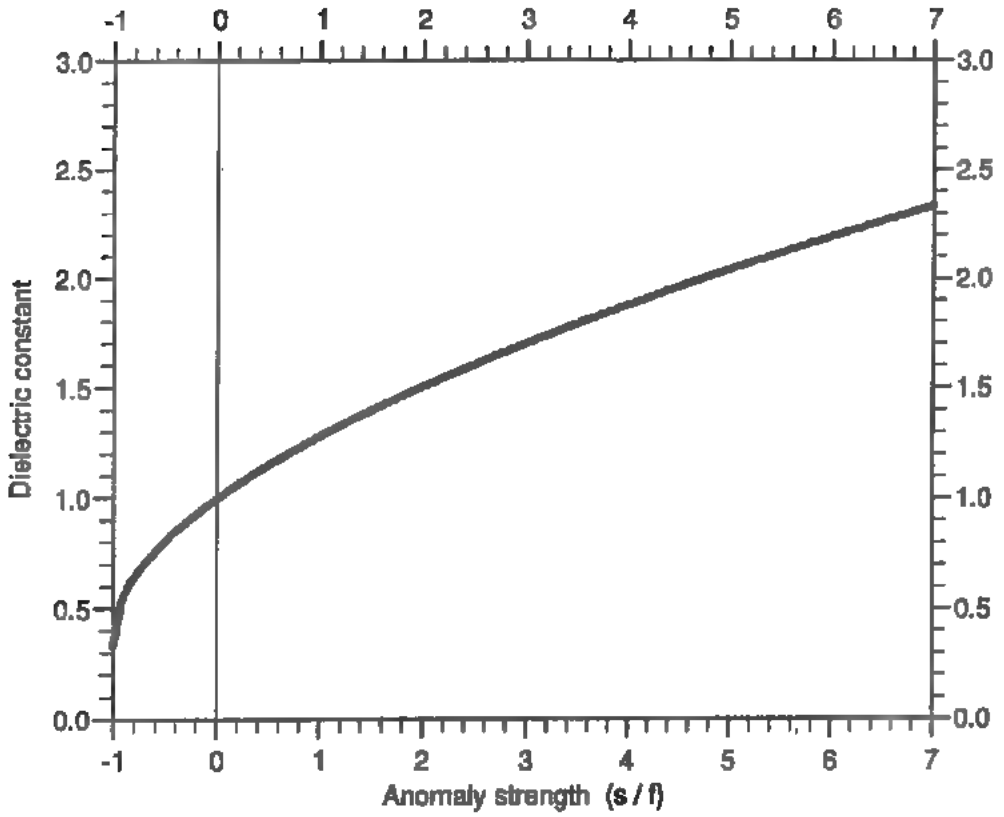
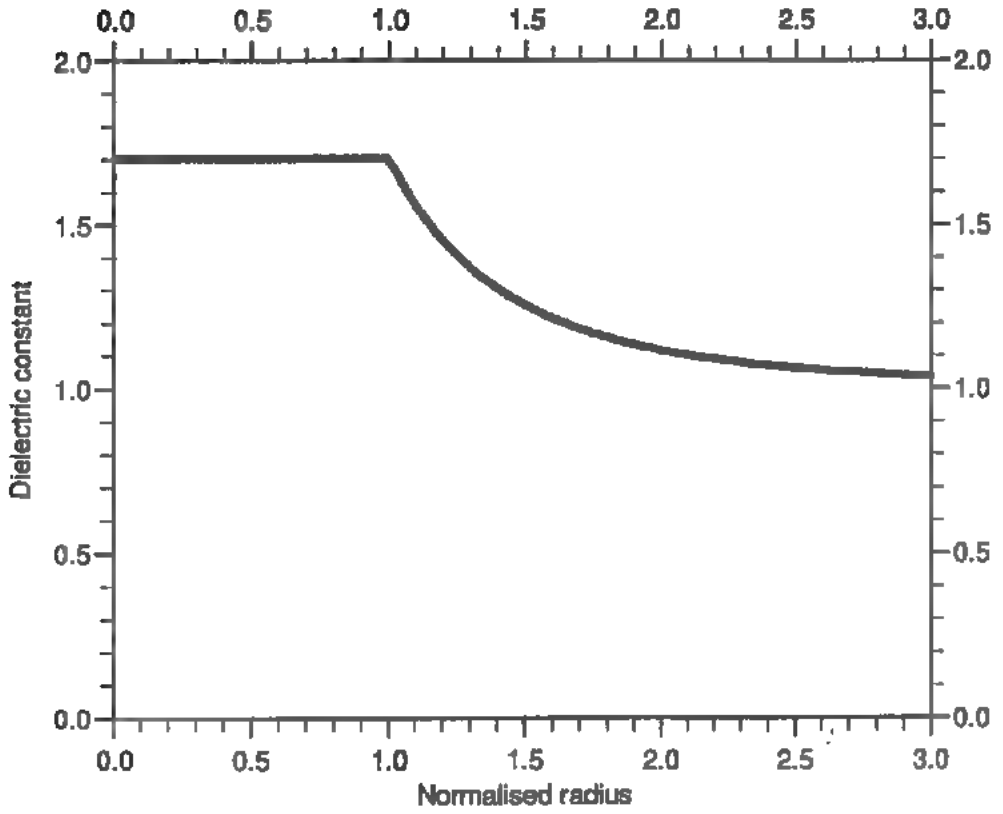
FIG 2

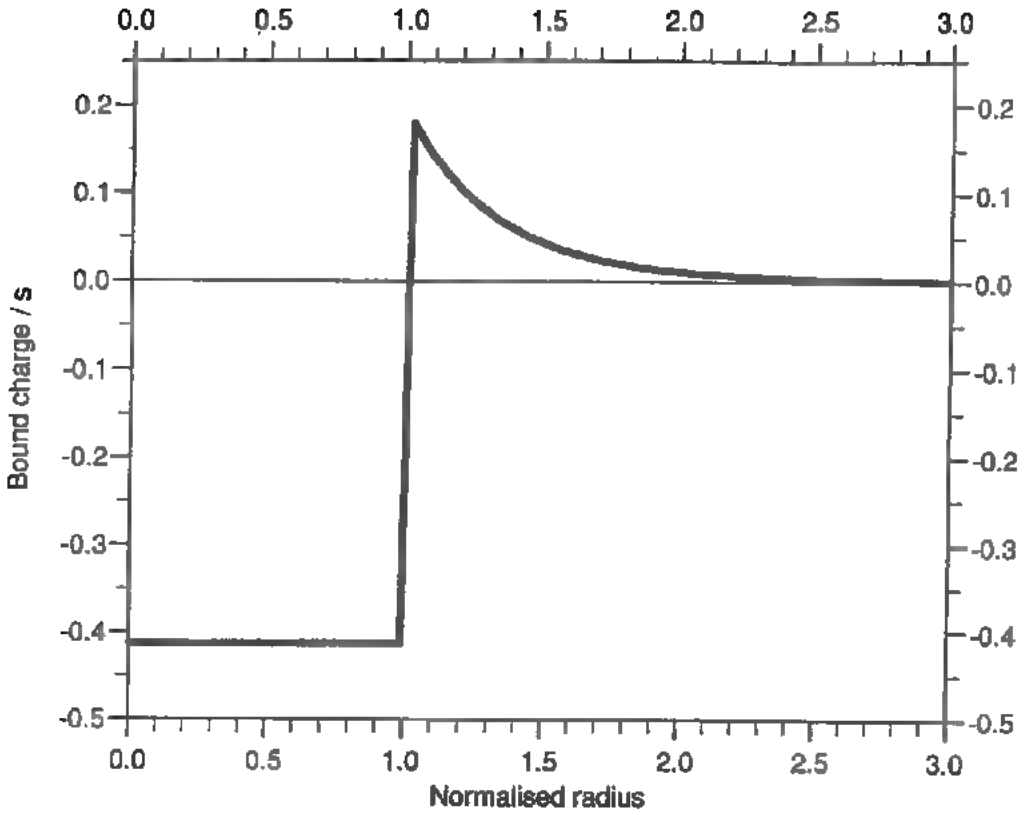



(a) Dielectric constant: r < b

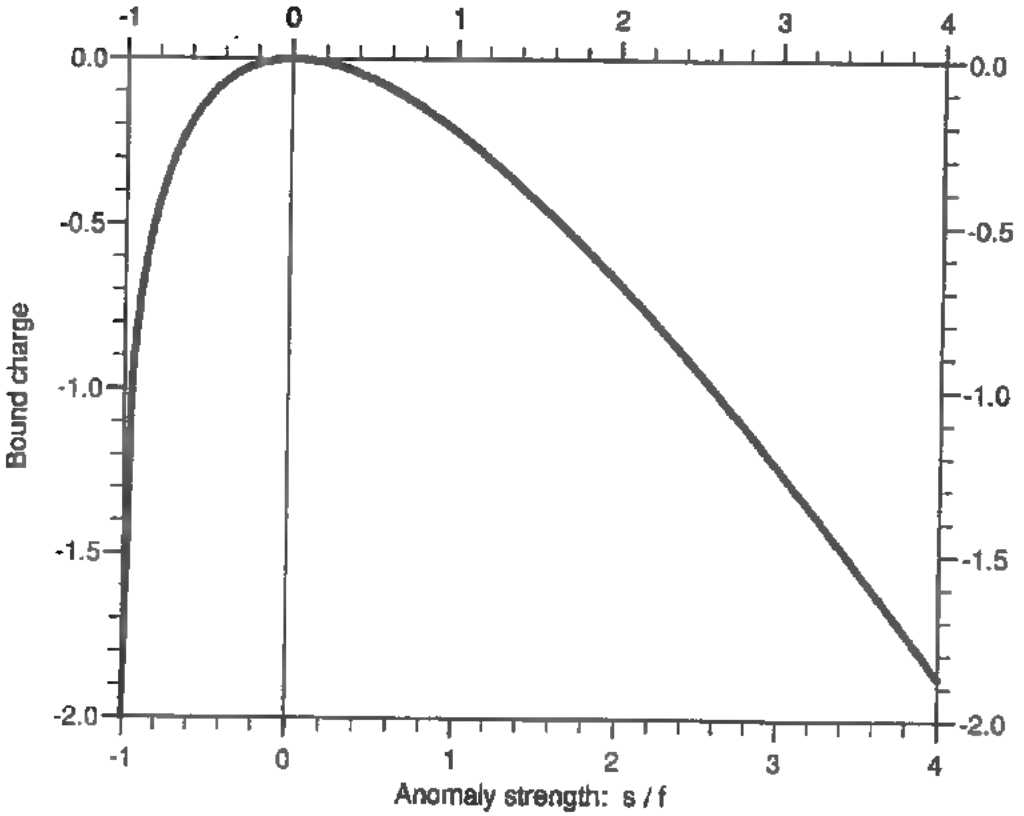



(b) Dielectric constant: s = 3 f

FIG 3

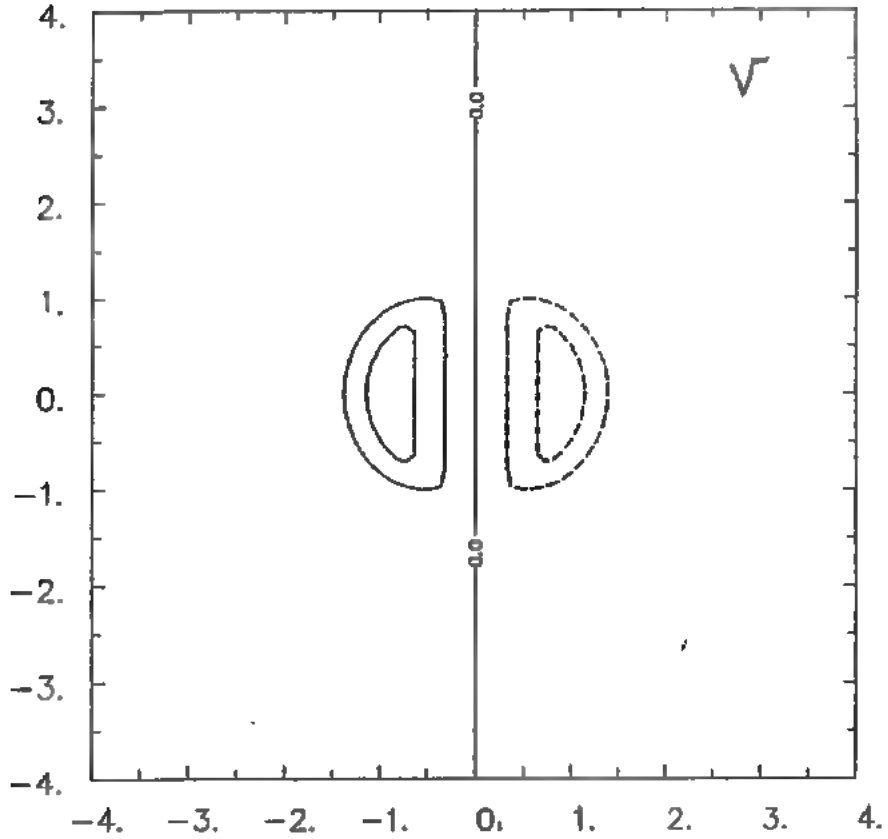



(a) Bound charge: s = 3 f




(b) Bound charge: r < b

FIG. 4



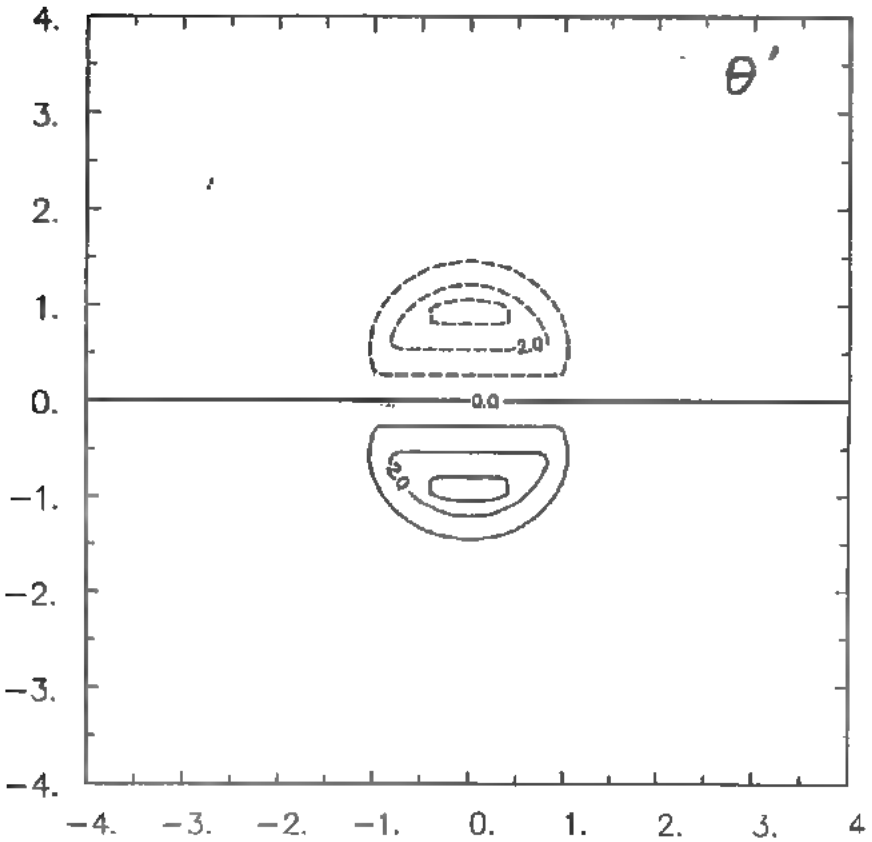




FIG. 5

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